Title: Estimating the Dimension of a Coastline

Brief Overview:

Students will learn the relationship between the size of the measuring unit and the measure of one, two, and three-dimensional figures. They will then extend this concept to non-integral dimension of an ideal mathematical fractal. Students will use this idea to estimate the dimension of a coastline.

NCTM 2000 Principles for School Mathematics:

- **Equity:** *Excellence in mathematics education requires equity - high expectations and strong support for all students.*

- **Curriculum:** *A curriculum is more than a collection of activities: it must be coherent, focused on important mathematics, and well articulated across the grades.*

- **Teaching:** *Effective mathematics teaching requires understanding what students know and need to learn and then challenging and supporting them to learn it well.*

- **Learning:** *Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge.*

- **Assessment:** *Assessment should support the learning of important mathematics and furnish useful information to both teachers and students.*

- **Technology:** *Technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning.*

Links to NCTM 2000 Standards:

- **Content Standards**

  Algebra
  Students will solve exponential equations to estimate the non-integral dimension of a fractal.

  Geometry
  Students will study the properties of fractal figures.

  Measurement
  Students will learn how the dimension of a figure affects the relationship between unit size and the measurement of both non-fractal and fractal figures.

- **Process Standards**

  <u>Reasoning and Proof</u>
  Students will generalize the idea of dimension to non-integral values.

  <u>Communication</u>
  Students will explain relevant terms and concepts, as well as their findings, both orally and in writing.

  <u>Connections</u>
  This topic connects geometry, measurement, and algebra.

Links to Maryland High School Mathematics Core Learning Units:

Functions and Algebra

- **1.1.2**
  Students will solve exponential equations to estimate the dimension of fractal objects.

Geometry, Measurement, and Reasoning

- **2.3.2**
  Students will study the properties of fractal figures, and learn how the dimension of a figure affects the relationship between unit size and the measurement of non-fractal and fractal figures.

Grade/Level:

Grades 8-11

Duration/Length:

This unit will take approximately 2 days, depending on students' background.

Prerequisite Knowledge:

Students should have working knowledge of the following concepts:

- Perimeter, area, and volume
- Exponential equations
- Functions

Student Outcomes:

Students will:

- learn the relationship between unit size and measure of a figure for one, two, and three-dimensional figures.
- extend this concept of non-integral dimension to an ideal mathematical fractal.
- learn how to estimate the dimension of a natural fractal.

Materials/Resources/Printed Materials:

- Rulers
- Index cards
- Scientific calculators
- Photocopied map of local coastline or river for every two students
- Worksheets 1, 2, and 3 (included at the end of this unit)

Development/Procedures:

Pass out an index card to each student. Have each one mark a segment 1 ½ inches along one edge, ½ inch along another, and 1/6 inch along another. Explain that students will be using these as measuring units. Label them unit A, unit B, and unit C. Note that unit B is 1/3 as long as unit A, so we will say that it represents a "scale factor" of 3. Similarly, C represents a scale factor of 9.

Have students complete worksheet 1. Then they should compare answers. Go over answers as necessary: [Answers: **1.** 2, 6, 18, 2n, 3; **2.** 4, 36, 324, $4n^2$, 9 or 3^2; **3.** 4, 108, 2916, $4n^3$, 27 or 3^3; 4. n^d, s^d.] The important conclusion is that the dimension is the exponent in the expression for measure as a function of scale factor.

Next have the students complete worksheet 2 to use this idea to determine the dimension of a famous fractal, the Koch snowflake. Again, have them compare answers, and discuss them as a class as necessary. [Answers: 12, 48, 192, 4]

Thus the dimension of the fractal must be the exponent of 3 that yields 4; that is, the solution to $3^d = 4$. Have students solve this equation using a calculator. If they have not learned logarithms, they may do it by trial and error. [Answer: (log 4)/(log 3) = approx. 1.26] Have them check this by testing the formula $l = 12\, n^{1.26}$ where l = length of boundary and n = scale factor.

Pass out photocopies of a local coastline or river, and have students work in pairs to complete worksheet 3. Note that the number by which the coastline's length is multiplied when changing from unit A to unit B may not be the same as it is when changing from unit B to unit C. The more similar these two numbers are, the more accurately the idea of fractal describes the coastline. Students will probably need to average these two factors to answer question 2. Calling that answer f, they will need to solve the equation $3^d = f$ for d in order to estimate the dimension. Average the students' results for number 4, and use that to do the check.

Assessment:

Have students (individually or in pairs) apply the procedure they have learned to estimate the dimension of another natural fractal. This could be another shoreline, river, or creek for which you have maps or a rough natural object (such as a leaf) which can be traced onto paper for easy measurement. Have students write a paper describing the procedure they used, its rationale, and the results they obtained.

A rubric can be used to evaluate the following aspects of the paper:

1. Description of mathematical concepts involved
2. Explanation of procedures used.
3. Description of results.
4. Use of notation and terminology.

Each of these aspects can be scored as follows:

4	Accurate and complete
3	Almost accurate and complete; minor errors made or details omitted
2	Work shows general understanding, but notable gaps or errors are present
1	Minimal or incomplete understanding, little chain of reasoning
0	Work is incorrect or meaningless; no evidence of understanding

Extension/Follow Up:

Students may wish to search for natural fractals on their own, and determine their dimensions. They can compare dimensions for several different objects or coastlines, and order these objects by their dimensions, thus developing a feel for dimension as a measure of roughness.

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Worksheet 1 - Investigating the Relationship Between Unit Size and Measure

1. One dimension: Draw a line segment 3 inches long below.

Measure your segment with each of the units you created on the index card. Record your results in the table below. Also note the pattern in the results you got, and write an expression for the length of the segment using a unit with scale factor n.

Measuring unit	A (scale factor 1)	B (scale factor 3)	C (scale factor 9)	Scale Factor n
Length				

Each time the scale factor is multiplied by 3, the length is multiplied by ______.

2. Two dimensions: Draw a rectangular region 6 inches long and 1 ½ inches wide.

Measure your rectangle with each of the units you created on the index card, and determine the area of the rectangular region. Record your results in the table below. Also note the pattern in the results you got, and write an expression for the area using a unit with scale factor n.

Measuring unit	A (scale factor 1)	B (scale factor 3)	C (scale factor 9)	Scale Factor n
Area				

Each time the scale factor is multiplied by 3, the area is multiplied by _____.

3. Three dimensions: Imagine (draw if necessary) a rectangular prism (box) with length 3 inches, width 1 ½ inches, and height 3 inches. Determine its volume using each unit on the index card, and record your results in the table below. Write an expression for the volume in terms of the scale factor n.

Measuring unit	A (scale factor 1)	B (scale factor 3)	C (scale factor 9)	Scale Factor n
Volume				

Each time the scale factor is multiplied by 3, the volume is multiplied by _____.

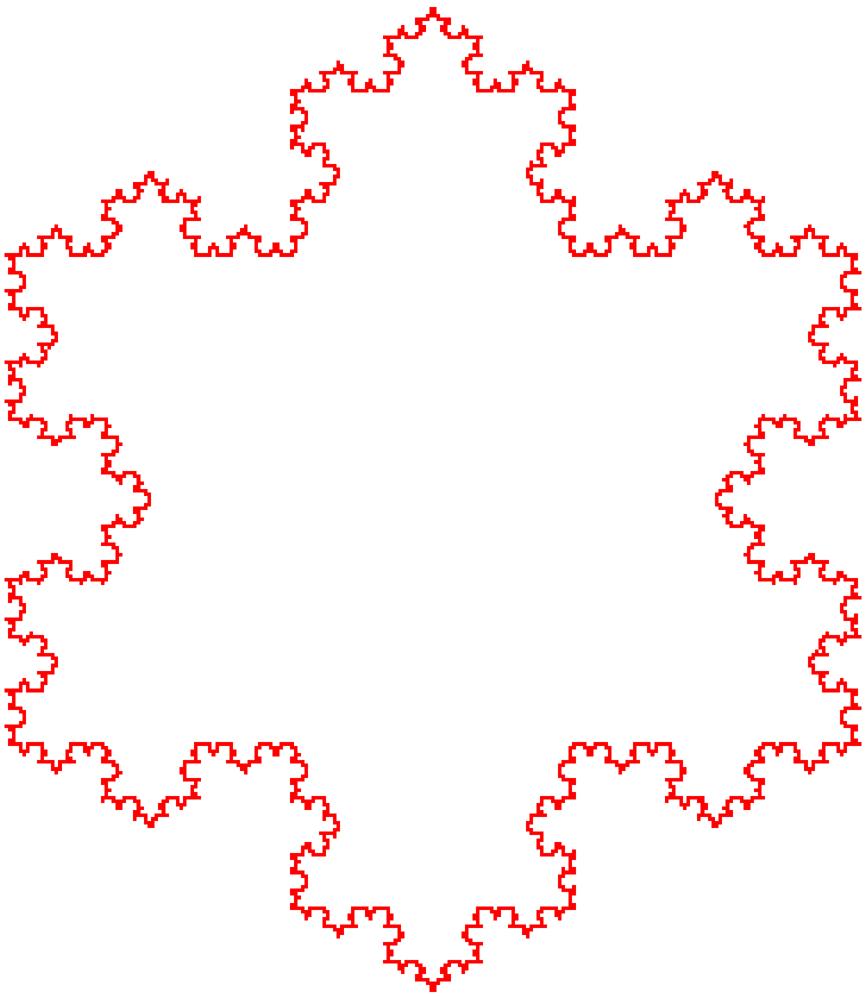
4. Generalize: For a d-dimensional object, its measure (length/area/volume) using a unit with scale factor n is equal to its measure using unit A multiplied by _______. That is, each time the scale factor is multiplied by s, the measure is multiplied by _______.

Worksheet 2 - Finding the Dimension of the Koch Snowflake

The Koch snowflake consists of the boundary drawn below. Measure it using the units on your index card, and complete the table below. Start at a convenient vertex, say the upper-most point, and work your way around the figure. Measure only as accurately as the unit allows; that is, do not try to measure fractions of units for the parts of the boundary that veer away from the edge of your index card.

Measuring unit	A (scale factor 1)	B (scale factor 3)	C (Scale Factor 9)
Length of boundary			

Each time the scale factor is multiplied by 3, the length is multiplied by _____.



Worksheet 3 - Estimating the Dimension of a Coastline

1. Measure the length of coastline or river using the units on your index card, and complete the table below. Measure only as accurately as the unit allows; that is, do not try to measure fractions of units for the parts of the coastline that veer away from the edge of your index card.

Measuring unit	A (scale factor 1)	B (scale factor 3)	C (scale factor 9)
Length of boundary			

2. Each time the scale factor is multiplied by 3, the length is multiplied by about _____.

3. What equation must be solved to estimate the dimension of the coastline or river?

4. Solve that equation.

5. Check your solution by writing an equation for the length as a function of the scale factor, and testing that equation against the data in your table.